11.1 Point on an Infinite Branch

A point P(x, y) on a curve $y = f(x)$ is said to tend to infinity if either or both the coordinates of P(x, y) tend to $\pm \infty$. In such a case it is said that the point P lies on an infinite branch of the curve [see **Fig. 11.1**].

■ **Illustration-1** : Consider the equation of an ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$. This curve lies wholly within a finite region in xy-plane. Thus it does not have an infinite branch. But if we consider the equation of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$, we know that it has two different branches, each of which is an infinite branch.

11.2 Definition of an Asymptote

If there exists a straight line at a finite distance from the origin such that the perpendicular distance of any point P on a given curve to the line tends to zero, as P tends to infinity along the curve, then the straight line is called an **asymptote** to the curve. It is also called a rectilinear asymptote (Refer **Fig. 11.1**).

■ **Illustration-2** : For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the straight lines $y = \pm \frac{b}{a}x$ are **two asymptotes** as seen in **Fig. 11.2**.

● **Remark** : It may be said that if a straight line cuts a curve in two points at an infinite distance from the origin and still does not lie at infinity wholly the straight line is called an asymptote to the given curve.

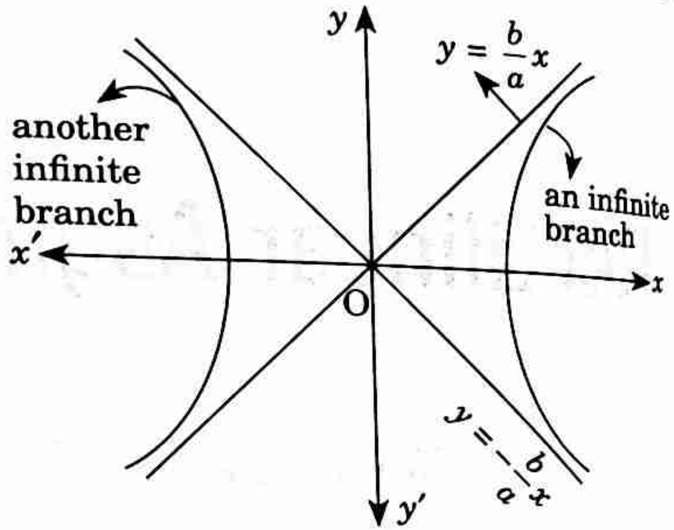

Fig. 11.2

11.3 Condition for $y = mx + c$ be an Asymptote

Let $y = mx + c$ (m, c : finite) be an equation of a straight line. Also let P(x, y) be a point on an infinite branch of a given curve $y = f(x)$.

Let p be perpendicular distance of P (x, y) from the straight line $y = mx + c$ (see **Fig. 11.1**)

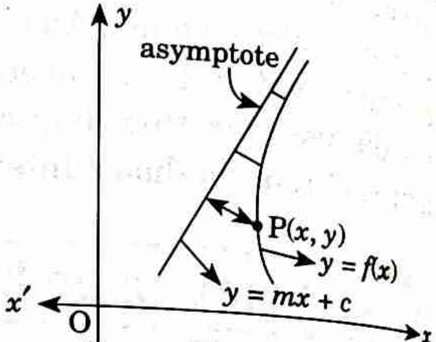

Fig. 11.1

$$\Rightarrow p = \frac{y - mx - c}{\sqrt{1+m^2}}$$

For $y = mx + c$ to be an asymptote to the given curve, we must have $p \to 0$, as $x \to \infty$ (along the curve)

$$\Rightarrow \lim_{x\to\infty}\left(\frac{y - mx - c}{\sqrt{1+m^2}}\right) = 0 \Rightarrow \boxed{\lim_{x\to\infty}(y - mx) = c} \quad \text{...(1)}$$

Let $y - mx = c + u$, so that $u \to 0$, as $x \to \infty$.

Thus, $\frac{y}{x} - m = \frac{c+u}{x} \Rightarrow \lim_{x\to\infty}\left(\frac{y}{x} - m\right) = \lim_{x\to 0}\left(\frac{c+u}{x}\right) = 0$

$$\Rightarrow \boxed{\lim_{x\to\infty}\left(\frac{y}{x}\right) = m} \quad \text{... (2)}$$

Therefore, $y = mx + c$ will be an asymptote to a curve $y = f(x)$ if conditions (1) and (2) hold.

■ **Illustration-3** : Consider the equation of the curve as $y = ax + b + \frac{d}{x}$ (a, b, $d \neq 0$). For $y = mx + c$ to be an asymptote of the curve, we have

$$m = \lim_{x\to\infty}\left(\frac{y}{x}\right) = \lim_{x\to\infty}\left(a + \frac{b}{x} + \frac{d}{x^2}\right) = a, \text{ and } c = \lim_{x\to\infty}(y - mx) = \lim_{x\to\infty}(y - ax) \text{ [since } m = a]$$

$$= \lim_{x\to\infty}\left(b + \frac{d}{x}\right) = b$$

Therefore, $y = ax + b$ be an asymptote of the given curve [since $m = a$ and $c = b$].

11.4 Rectangular and Oblique Asymptotes

An asymptote may be parallel to x-axis (called a horizontal asymptote), or it may be parallel to y-axis (called a vertical asymptote), otherwise it is called an **oblique asymptote** (No parallel to either axis). If an asymptote is parallel to either axis, then it is called a **rectangular asymptote**. [see **Fig. 11.3** and **Fig. 11.4**]

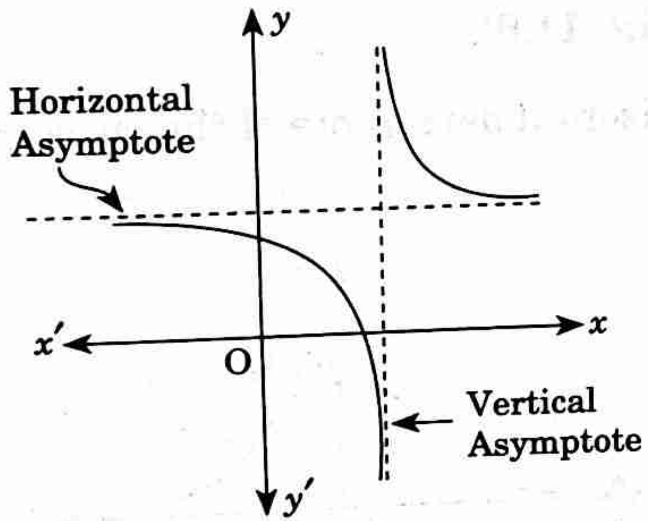

Fig. 11.3 (Rectangular Asymptote)

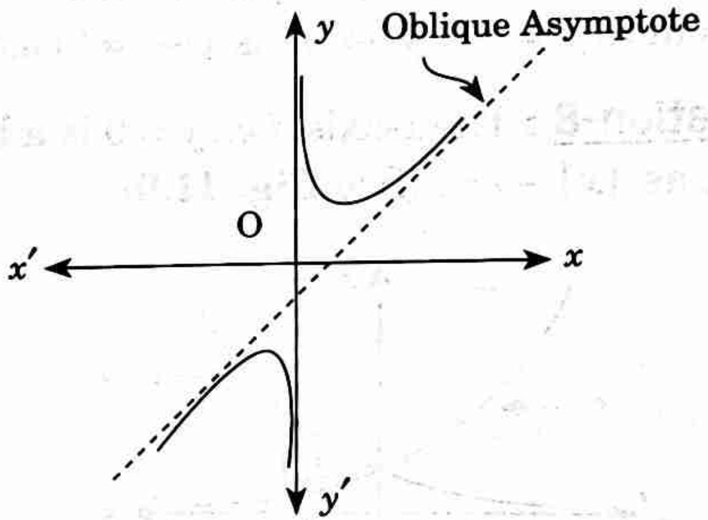

Fig. 11.4 (Oblique Asymptote)

- **Asymptote Parallel to x-axis :** The necessary and sufficient condition of the horizontal line $y = b$ to be an asymptote to the curve $x = \phi(y)$ is that $|\phi(y)| \to \infty$, as $y \to b$.
- **Asymptote Parallel to y-axis :** The necessary and sufficient condition of the vertical line $x = a$ to be an asymptote to the curve $y = f(x)$ is that $|f(x)| \to \infty$, as $x \to a$.

■ **Illustration-4** : For $y = \tan x$, we have $|y| \to \infty$, as $x \to \pm \frac{\pi}{2}$. Thus $x = \pm \frac{\pi}{2}$ are the two asymptotes. Similarly, $x = \pm (2n + 1)\frac{\pi}{2}$ are all asymptotes of $y = \tan x (n \in \mathbb{Z})$. See **Fig. 11.5** for vertical asymptotes of $y = \tan x$.

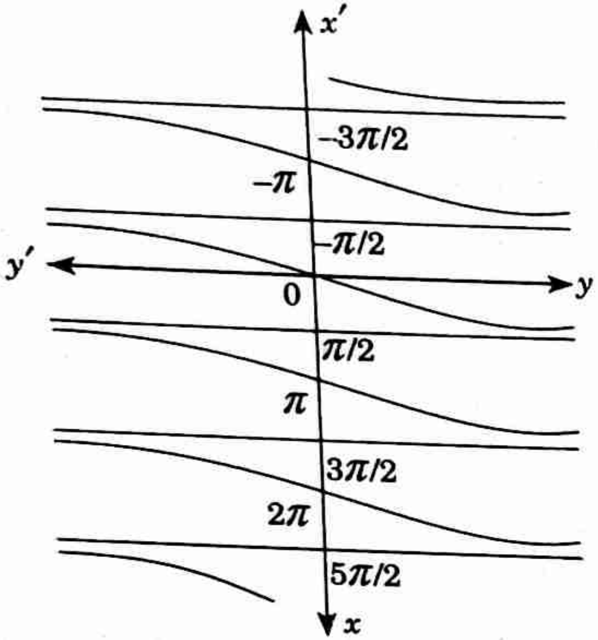

Fig. 11.5

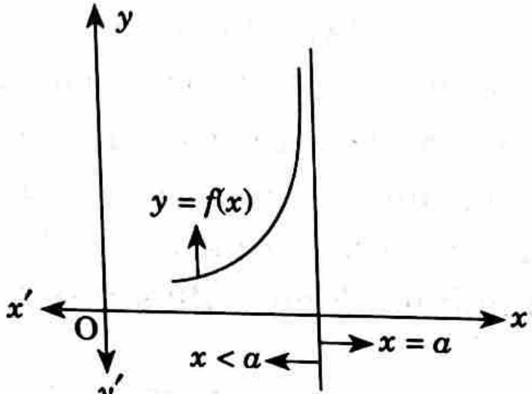

Fig. 11.6

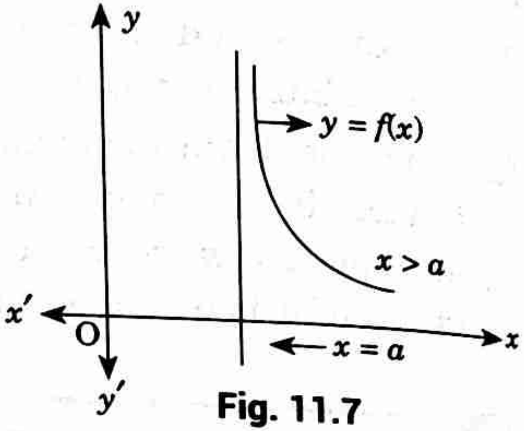

Fig. 11.7

■ **Illustration-7** : For the curve $y = e^x$, $y \to 0$, as $x \to -\infty$. Hence $y = 0$ is an asymptote, which is a horizontal asymptote. On the same way $y = 0$ is also a horizontal asymptote of the curve $y = e^{-x}$ ($y \to 0$, as $x \to \infty$) [see **Fig. 11.8**].

■ **Illustration-8** : The x-axis, *i.e.*, $y = 0$ is a horizontal asymptote of the curve $y = e^{-x^2}$. Here $y \to \infty$, as $|x| \to \infty$. [See **Fig. 11.9**]

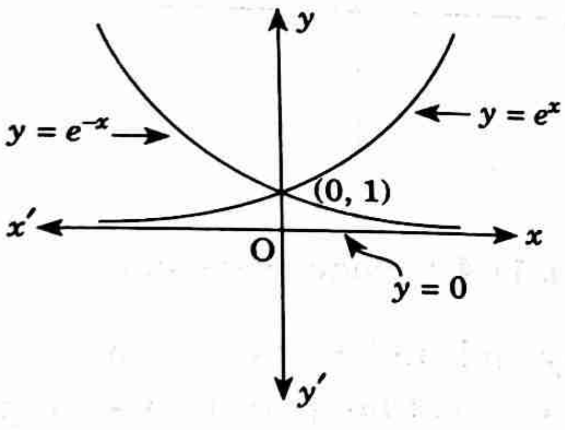

Fig. 11.8

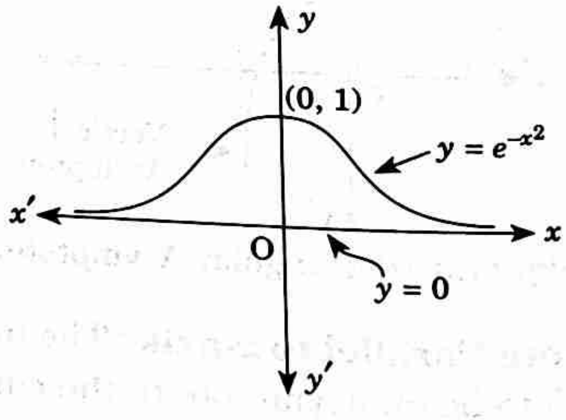

Fig. 11.9

11.5 Asymptotes Parallel to x-axis and y-axis for Rational Algebraic curve $f(x, y) = 0$

1. The real linear factors in the coefficeint of the highest power of x in the algebraic curve $f(x, y) = 0$ when equated to zero gives us asymptotes parallel to x-axis (*i.e.*, **horizontal asymptotes**).
2. The real linear factors in the coefficient of the highest power of y in the algebraic curve $f(x, y) = 0$ when equated to zero gives us asymptotes parallel to y-axis. (*i.e.*, **vertical asymptotes**).

vertical asymptote.

Illustration-9 : Consider the curve $x^3 - 2x^2y + xy^2 + x - xy + 2 = 0$. Here the coefficient of the highest power of x (*i.e.*, of x^3) is 1, a constant. Hence the curve has no asymptote parallel to x-axis.

Again the coefficient of the highest power of y (*i.e.*, of y^2) is x. Thus $x = 0$ is an asymptote parallel to y-axis (here y-axis itself).

Illustration-10 : For the curve $y^3 + 2x^2 = 1$, there is neither a horizontal asymptote nor a vertical asymptote, as the coefficients of the highest powers of both x and y *i.e.*, of x^2 and y^3 are constants.

Illustration-11 : Consider the curve $x^4 + x^2y^2 - 9(y^2 + 9) = 0$. It has no horizontal asymptote as the coefficient of the highest power of x (*i.e.*, of x^4) is 1, a constant. But the coefficient of the highest power of y (*i.e.*, of y^2) is $x^2 - 9$. Thus $x^2 - 9 = 0$ or $x = \pm 3$ are two vertical asymptotes.

11.6 Rules for Finding out the Oblique Asymptotes

If the asymptotes of a curve $f(x, y) = 0$ are neither horizontal nor vertical, then those asymptotes are said to be oblique asymptotes [see **Fig. 11.4**]. Now, we describe here four different methods to determine the oblique asymptotes.

- **Method-1 :** Consider a general rational algebraic curve $f(x, y) = 0$ of degree n as

$$f(x, y) = (a_0 y^n + a_1 y^{n-1} x + a_2 y^{n-2} x^2 + \dots + a_{n-1} y x^{n-1} + a_n x^n) + (b_1 y^{n-1} + b_2 y^{n-2} x + \dots + b_{n-1} y x^{n-2} + b_n x^{n-1}) + \dots + (k_{n-1} y + k_n x) + l_n = 0, \quad \dots (1)$$

where $a_0, a_1, \dots, a_n, b_1, b_2, \dots, b_n, k_{n-1}, k_n$ and l_n are constants. After some manipulation (1) takes the form

$$x^n \phi_n\left(\frac{y}{x}\right) + x^{n-1} \phi_{n-1}\left(\frac{y}{x}\right) + \dots + x\phi_1\left(\frac{y}{x}\right) + \phi_0\left(\frac{y}{x}\right) = 0, \quad \dots (2)$$

where $\phi_j\left(\frac{y}{x}\right)$ is a polynomial in $\left(\frac{y}{x}\right)$ of degree j.

First we check whether (2) has horizontal or vertical asymptotes. If any such exists we find out those using the methods stated in **Art. 11.5**.

- **Steps for finding out oblique asymptotes :**

1. We put $x = 1$ and $y = m$ in $x^n \phi_n\left(\frac{y}{x}\right)$ [the highest degree term of (2)] and obtain $\phi_n(m)$. Let $m = m_1, m_2, m_3, \dots m_n$ are the n roots of the nth degree polynomial equation

$\phi_n(m) = 0$. ... (3)

We also find out $\phi_n'(m)$ [derivative of $\phi_n(m)$ with respect to m]

2. We again put $x = 1$ and $y = m$ in $x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)$ [the second highest degree term) of (2)] and obtain $\phi_{n-1}(m)$, a polynomial in m of degree $(n - 1)$.

3. For $m = m_1$, we find out $c = c_1 = \dfrac{-\phi_{n-1}(m_1)}{\phi_n'(m_1)}$, ... (4)

provided $\phi_n'(m_1) \neq 0$. With those values of m_1 and c_1, the straight line $y = m_1x + c_1$ is an asymptote of (1) or of (2). In a similar manner we can calculate other asymptotes like $y = m_2x + c_2, y = m_3x + c_3, \ldots, y = m_nx + c_n$.

4. If $\phi_n'(m) = 0$, for some m, we will have parallel asymptotes. We use the formula

$$\frac{c^2}{2}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$$

to calculate two values of c, viz. c_1 and c_2. Then $y = mx + c_1$ and $y = mx + c_2$ be the two parallel asymptotes, having same slope m.

■ **Illustration-12** : Consider the cubic $y^3 + x^2y + 2xy^2 - y + 1 = 0$. [CH 1984]

Here the coefficient of the highest power of x (viz. x^2) is y. It means $y = 0$ is a horizontal asymptote. Here $\phi_3(m) = m^3 + 2m^2 + m$ (by putting $x = 1$ and $y = m$ in the highest degree terms $y^3 + 2xy^2 + x^2y$), $\phi_3'(m) = 3m^2 + 4m + 1$. $\phi_3(m) = 0 \Rightarrow m = 0, -1, -1$.

Also $\phi_3''(m) = 6m + 4$.

$\phi_2(m) = 0$, for all m, as there is no second degree term in the given equation. $\phi_1(m) = -m$ (by putting $x = 1$ and $y = m$ in 1st degree term '$-y$').

As $m = -1$ occurs twice, we will have parallel asymptotes. Those can be found from

$$\frac{c^2}{2}\phi_3''(-1) + c\phi_2'(-1) + \phi_1(-1) = 0$$

$$\Rightarrow \frac{c^2}{2}[6\cdot(-1) + 4)] + c\cdot 0 - (-1) = 0$$

$\Rightarrow c = \pm 1$. Hence $y = -x \pm 1$ are also two asymptotes.

(for $m = 2$ and $c = 1$) and $y = 2x + 2$ (for m ...

Method-3 : We recall the equation (1) of a rational algebraic curve of degree n of the form $f(x, y) = 0$.

I. Let $(a_1x + b_1y + c_1)$ be a non-repeated factor of the n-th degree terms of $f(x, y) = 0$. So equation (1) now may be expressed as

$$(a_1x + b_1y + c_1)F_{n-1} + R_{n-1} = 0, \quad \text{... (9)}$$

where F_{n-1} contains terms of degree $(n - 1)$ and R_{n-1} contains terms of degree $\leq (n - 1)$.

The asymptote of (9) is given by

$$(a_1x + b_1y + c_1) + \lim_{|x|\to\infty} \frac{R_{n-1}}{F_{n-1}} = 0, \quad \left[\text{where } \lim_{|x|\to\infty}\left(\frac{y}{x}\right) = -\frac{a_1}{b_1}\right] \quad \cdots (10)$$

For other non-repeated factors (if any such) of (1), we may proceed similarly.

II. Let equation of the curve (1), *i.e.*, $f(x, y) = 0$ be expressed as

$$(a_1x + b_1y + c_1)^2F_{n-2} + (a_1x + b_1y + c_1)R_{n-2} + T_{n-2} = 0, \quad \cdots (11)$$

where F_{n-2} contains terms of degree $(n-2)$, R_{n-2} contains terms of degree $(n-2)$ and T_{n-2} contains terms of degree $\le (n-2)$. In a similar way the two parallel asymptotes of (11) (parallel to $a_1x + b_1y + c_1 = 0$) are given by

$$(a_1x + b_1y + c_1)^2 + (a_1x + b_1y + c_1) \times \lim_{|x|\to\infty} \frac{R_{n-2}}{F_{n-2}} + \lim_{|x|\to\infty} \frac{T_{n-2}}{F_{n-2}} = 0, \quad \cdots (12)$$

where $\lim_{|x|\to\infty}\left(\frac{y}{x}\right) = -\frac{a_1}{b_1}$.

■ **Illustration-15** : Let $f(x, y) = (x + y)(x - 2y)(x - y)^2 + 3xy(x - y) + x^2 = 0$ [CH 2006]

Using (10), the asymptote parallel to $x + y = 0$ is given by

$$(x + y) + \lim_{\substack{|x|\to\infty \\ \frac{y}{x}=-1}} \frac{3xy(x-y)+x^2}{(x-2y)(x-y)^2} = 0 \text{ or } (x + y) + \lim_{\substack{|x|\to\infty \\ \frac{y}{x}=-1}} \frac{3\cdot 1\cdot\left(\frac{y}{x}\right)\left(1-\frac{y}{x}\right)+\frac{1}{x}}{\left(1-2\cdot\frac{y}{x}\right)\left(1-\frac{y}{x}\right)^2} = 0$$

or, $x + y - \dfrac{1}{2} = 0$. Similarly the asymptote parallel to $x - 2y = 0$ is given by

$$(x - 2y) + \lim_{\substack{|x|\to\infty \\ \frac{y}{x}=\frac{1}{2}}} \frac{3xy(x-y)+x^2}{(x+y)(x-y)^2} = 0 \text{ or, } x - 2y + 1 = 0.$$

Again using (12), two parallel asymptotes (parallel to $x - y = 0$) are given by

$$(x - y)^2 + (x - y) \times \lim_{\substack{|x|\to\infty \\ \frac{y}{x}=1}} \frac{3xy}{(x+y)(x-2y)} + \lim_{\substack{|x|\to\infty \\ \frac{y}{x}=1}} \frac{x^2}{(x+y)(x-2y)} = 0$$

or, $(x - y)^2 + (x - y) \times \left(-\dfrac{3}{2}\right) + \left(-\dfrac{1}{2}\right) = 0$

or, $2(x - y)^2 - 3(x - y) - 1 = 0$ or, $x - y = \dfrac{3 \pm \sqrt{17}}{4}$.

■ **Illustration-16** : Consider the Folium of Descartes : $x^3 + y^3 = 3axy$. It may be expressed as $(x + y)(x^2 - xy + y^2) - 3axy = 0$.

The given curve has no horizontal or vertical asymptotes, as the coefficient of the highest powers of x and y (viz. x^3 and y^3) are both constants (here 1). So the only possible asymptote (parallel to $x + y = 0$) is given by

$$(x + y) - \lim_{\substack{|x|\to\infty \\ \frac{y}{x}=-1}} \frac{3axy}{x^2 - xy + y^2} = 0 \quad \text{[using method 3]}$$

or, $(x + y) - \displaystyle\lim_{\substack{|x|\to\infty \\ \frac{y}{x}=-1}} \frac{3a\cdot 1\cdot\left(\frac{y}{x}\right)}{1-\frac{y}{x}+\left(\frac{y}{x}\right)^2} = 0$ or, $x + y - \dfrac{(-3a)}{3} = 0$ or, $x + y + a = 0$.

Folium of Descartes ... But it has only one asymptote.

Method-4 (By Inspection only) : If the equation of the curve $f(x, y) = 0$ can be expressed in the form

$$F_n + F_{n-2} = 0, \quad \text{... (13)}$$

where, F_n is a polynomial in x and y of degree n and can be broken into n distinct real linear factors and F_{n-2} is a polynomial in x and y of degree $\leq n - 2$, then $F_n = 0$ gives the equations of n asymptotes (assuming no two of which are parallel).

Note

The difference of degrees of the polynomials F_n and F_{n-2} is at least two.

Illustration-17 : The equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ may be expressed as $\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right) - 1$, which is of the form $F_2 + F_0 = 0$. Here F_2 and F_0 are polynomials of degree 2 and 0 respectively. Moreover F_2 is broken into two non-repeated real linear factors. Thus $F_2 = 0$ gives the equations of two asymptotes. They are $\frac{x}{a}+\frac{y}{b} = 0$ and $\frac{x}{a}-\frac{y}{b} = 0$.

Illustration-18 : Let the equation of the given curve be $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$. It can be written as $F_3 + F_1 = 0$, **[CH 1998, 2007; NBH 1997, 2006]** where $F_3 = x^3 - 6x^2y + 11xy^2 - 6y^3 = (x - y)(x - 2y)(x - 3y)$ and $F_1 = x + y + 1$. As F_3 contains distinct real linear factors and difference of degrees in F_3 and F_1 is two, $F_3 = 0$ gives the asymptotes. Thus $x - y = 0$, $x - 2y = 0$ and $x - 3y = 0$ are the equations of three asymptotes of the given curve.

Illustration-19 : For the cubic $(x - y + 2)(2x - 3y + 4)(4x - 5y + 6) + (5x - 6y + 7) = 0$, the asymptotes are clearly $x - y + 2 = 0$, $2x - 3y + 4 = 0$ and $4x - 5y + 6 = 0$.

11.7 Intersection of a Curve with its Asymptotes

We have seen in (3) of **Art. 11.6** that the slopes (m) of asymptotes (not parallel to y-axis) for a rational algebraic curve $f(x, y) = 0$ of degree n are given as $\phi_n(m) = 0$. It gives n values of m at most. For each value of 'm', we have a corresponding 'c' given in (4) as $c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}$ $[\phi_n'(m) \neq 0]$.

In this way we have n number of asymptotes of the form : $y = mx + c$ of a rational algebraic curve of degree n. So we can conclude that **a rational algebraic curve of degree n has n number of asymptotes, in general**. We have also observed in Method 4 of **Art. 11.6** that the equations of the curve (of degree n) and its n asymptotes are respectively given by

$$F_n + F_{n-2} = 0 \text{ and } F_n = 0.$$

Their intersections are therefore given by $F_{n-2} = 0$. In other words, the n asymptotes ($F_n = 0$) intersect the curve ($F_n + F_{n-2} = 0$) again at points lying upon the curve $F_{n-2} = 0$. We know each asymptote cuts its curve at two points at infinity, and hence cuts at $(n - 2)$ points at a finite distance. So there are altogether $n(n - 2)$ number of points of intersection of n asymptotes with its curve at a finite distance. These $n(n - 2)$ points lie on a certain curve, $F_{n-2} = 0$, of degree $(n - 2)$.

Notes

1. The asymptotes of a cubic (degree 3) will cut the curve again in 3(3 – 2), *i.e.*, in 3 points lying on a straight line.
2. The asymptotes of quartic curve (degree 4) will cut the curve again in 4(4 – 2), *i.e.*, in 8 points lying on a conic section.

Illustration-20 : The asymptotes of the cubic $x(x^2 - y^2) + y(3y - x) = 0$ may be found out by using any method between Method-1—Method-3 stated earlier. **[BH 2006; CH 2014]**

The asymptotes of this cubic are $x - 3 = 0$, $x + y + 2 = 0$ and $x - y + 1 = 0$. All these asymptotes can be written together as $(x - 3)(x + y + 2)(x - y + 1) = 0$, *i.e.*, $x(x^2 - y^2) + y(3y - x) - 7x + 3y - 6 = 0$. The equation of the given curve now can be expressed as

$$[x(x^2 - y^2) + y(3y - x) - 7x + 3y - 6] + (7x - 3y + 6) = 0,$$

i.e., $F_3 + F_1 = 0$. (difference in degrees in F_3 and F_1 is 2),

where $F_3 = 0$ gives the equations of the asymptotes and $F_1 = 7x - 3y + 6 = 0$ is an equation of a straight line. Thus, the number of points of intersection of the curve with its asymptotes is 3(3 – 2), *i.e.*, 3. These there points of intersection lie on the straight line $7x - 3y + 6 = 0$.

There does not ...

2. **Find the asymptotes of $x^2y^2 = a^2(x^2 + y^2)$.** **[CU 2013]**

Solution Here the equation of the curve is $x^2y^2 = a^2(x^2 + y^2)$ ... (1)

(1) can be written as $x^2y^2 - a^2x^2 - a^2y^2 = 0$.

The degree of the equation is four.

So (1) has at most four asymptotes.

The highest degree term in x is x^2 and the coefficient of x^2 is $y^2 - a^2$. Therefore we get the horizontal asymptotes of the curve by equating $y^2 - a^2$ to zero.

Therefore two horizontal asymptotes are

$$y = \pm a \quad \text{... (2)}$$

Again the highest degree term in y is y^2 and the coefficient of y^2 is $(x^2 - a^2)$.

Therefore, $x^2 - a^2 = 0$ gives us two vertical asymptotes and they are $x = \pm a$ ... (3)

Since degree of the given curve is 4, and we have already got four asymptotes, so, the given curve has no oblique asymptotes.

Thus, the required asymptotes are $x = \pm a$, $y = \pm a$.

Thus, ...

3. **Find the rectilinear asymptotes of**

(i) $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 - 1 = 0$ **[CU 2011, 2015]**

(ii) $x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0$ **[CU 2014, 2016]**

(iii) $xy^2 - x^2y = x + y + 1.$ **[CU 2008, 2011, 2014]**

Solution (i) Clearly, the given curve

$$y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 - 1 = 0 \quad \text{... (1)}$$

has no horizontal or vertical asymptote.

For oblique asymptotes, we factorise the third degree terms of the equation as

$$y^3 - xy^2 - x^2y + x^3 = y^2(y - x) - x^2(y - x)$$
$$= (y - x)(y^2 - x^2) = (y - x)(y - x)(y + x) = (y - x)^2(y + x).$$

Therefore, there are three oblique asymptotes of the curve, two are parallel to $y - x = 0$ and the other is parallel to $y + x = 0$.

The asymptotes parallel to $y - x = 0$ are given by

$$(y - x)^2 - (y - x)\lim_{\substack{|x|\to\infty \\ y = x}} \frac{y + x}{y + x} + \lim_{\substack{|x|\to\infty \\ y = x}} \frac{-1}{y + x} = 0 \quad \text{or, } (y - x)^2 - (y - x)\lim_{|x|\to\infty} \frac{2x}{2x} + \lim_{|x|\to\infty} \frac{-1}{2x} = 0$$

or, $(y - x)^2 - (y - x) + 0 = 0$ or, $(y - x)(y - x - 1) = 0$

i.e., $y - x = 0$ and $y - x - 1 = 0$.

The asymptote parallel to $y + x = 0$ is given by

$$(y + x) + (y + x)\lim_{\substack{|x|\to\infty \\ y = -x}} \frac{x - y}{(y - x)^2} + \lim_{\substack{|x|\to\infty \\ y = -x}} \frac{-1}{(y - x)^2} = 0$$

or, $(y+x)+(y+x)\lim\limits_{|x|\to\infty}\frac{2x}{4x^2}+\lim\limits_{|x|\to\infty}\frac{-1}{4x^2}=0$ or, $(y+x)+0+0=0$ *i.e.*, $y+x=0$.

Therefore, the required three asymptotes are $y-x=0$, $y-x-1=0$ and $y+x=0$.

Note

The given equation can be written as $(y-x)^2(y+x)-(y^2-x^2)-1=0$

or, $(y+x)\{(y-x)^2-(y-x)\}-1=0$ or, $(y+x)(y-x)(y-x-1)-1=0$.

This is of the form $F_3+F_1=0$.

Therefore required asymptotes are $y+x=0$, $y-x=0$ and $y-x-1=0$.

(ii) Equation of the curve is $x^3-2y^3+xy(2x-y)+y(x-y)+1=0$

or, $x^3+2x^2y-xy^2-2y^3+y(x-y)+1=0$. ... (1)

Clearly, (1) is a 3rd degree equation in x and y.

The terms involving x^3 and y^3 are present in the equation.

So the given curve has no horizontal or vertical asymptotes.

For oblique asymptotes, we factorise the third degree terms of the equation as,

$x^3+2x^2y-xy^2-2y^3=x^2(x+2y)-y^2(x+2y)=(x+2y)(x^2-y^2)=(x+2y)(x-y)(x+y)$

Therefore, there are three oblique asymptotes which are parallel to $x+2y=0$, $x-y=0$ and $x+y=0$.

The asymptote parallel to $x+2y=0$ is given by

$$x+2y+\lim_{\substack{|x|\to\infty\\ y=-\frac{x}{2}}}\frac{y(x-y)+1}{(x-y)(x+y)}=0 \quad \text{or, } x+2y+\lim_{|x|\to\infty}\frac{-\frac{x}{2}\left(x+\frac{x}{2}\right)+1}{\left(x+\frac{x}{2}\right)\left(x-\frac{x}{2}\right)}=0$$

or, $x+2y+\dfrac{-\frac{3}{4}}{\frac{3}{4}}=0$ or, $x+2y-1=0$.

The asymptote parallel to $x+y=0$, is given by $x+y+\lim\limits_{\substack{|x|\to\infty\\ y=-x}}\dfrac{y(x-y)+1}{(x-y)(x+2y)}=0$

or, $x+y+\lim\limits_{|x|\to\infty}\dfrac{-x(x+x)+1}{(x+x)(x-2x)}=0$ or, $x+y+\dfrac{-2}{-2}=0$ or, $x+y+1=0$.

The asymptote parallel to $(x-y)=0$, is given by

$$(x-y)+(x-y)\lim_{\substack{|x|\to\infty\\ y=x}}\frac{y}{(x+y)(x+2y)}+\lim_{\substack{|x|\to\infty\\ y=x}}\frac{1}{(x+y)(x+2y)}=0.$$

or, $(x-y)+(x-y)\lim\limits_{|x|\to\infty}\dfrac{x}{2x\cdot 3x}+\lim\limits_{|x|\to\infty}\dfrac{1}{2x\cdot 3x}=0$ or, $x-y=0$.

Therefore, the required three asymptotes are $x+2y-1=0$, $x+y+1=0$ and $x-y=0$.

(iii) The equation of the curve is $xy^2-x^2y=x+y+1$. ... (1)

(1) can be written as $xy(y-x)-(x+y+1)=0$. ... (2)

(2) is of the form $F_3+F_1=0$, where F_3 is a product of three non-repeated real linear factors and F_1 is of degree at most one.

So $F_3=0$ gives us the three required asymptotes and they are $x=0$, $y=0$ and $y-x=0$.

4. **Prove that a parabola has no asymptote.**

Solution Let the equation of the parabola be $y^2 = 4ax$, *i.e.*, $y^2 - 4ax = 0$. ... (1)

The degree of the equation is two.

So (1) has at most two asymptotes. Since the highest degree term in y *i.e.*, y^2 is present, so, the parabola has no asymptotes parallel to y-axis *i.e.*, vertical asymptotes.

Now for oblique and horizontal asymptotes, considering second and first degree terms (putting $x = 1$ and $y = m$), we get

$\phi_2(m) = m^2$, $\phi_1(m) = -4a$. So, $\phi'_2(m) = 2m$. Now $\phi_2(m) = 0$ gives $m = 0$.

Now $c = -\dfrac{\phi_1(m)}{\phi'_2(m)} = -\dfrac{-4a}{2m}$. For $m = 0$, $c = -\dfrac{-4a}{0} = \infty$.

So there exists no asymptote. Therefore, a parabola has no asymptote.

7. **Find the asymptotes (if any) of the curve $x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0.$**

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Solution Here the degree of the equation of the curve is four. So it has at most four asymptotes. Here x^4 and y^4 are present in the equation. So the curve has no horizontal or vertical asymptotes. For oblique asymptotes we factorise the fourth degree terms as

$$x^4 - y^4 = (x^2 - y^2)(x^2 + y^2) = (x - y)(x + y)(x^2 + y^2)$$

Therefore, there are two oblique asymptotes of the curve, which are parallel to

$x - y = 0$ and $x + y = 0$.

The asymptote parallel to $x - y = 0$ is given by

$$x - y + \lim_{\substack{|x|\to\infty \\ y=x}} \frac{3x^2y + 3xy^2 + xy}{(x+y)(x^2+y^2)} = 0 \quad \text{or, } x - y + \lim_{|x|\to\infty} \frac{3x^3 + 3x^3 + x^2}{2x(2x^2)} = 0$$

$$\text{or, } x - y + \lim_{|x|\to\infty} \frac{6x^3 + x^2}{4x^3} = 0 \text{ or, } x - y + \frac{3}{2} = 0 \quad \text{or, } 2x - 2y + 3 = 0.$$

The asymptote parallel to $x + y = 0$ is given by

$$x + y + 3(x + y) \lim_{\substack{|x|\to\infty \\ y=-x}} \frac{xy}{(x-y)(x^2+y^2)} + \lim_{\substack{|x|\to\infty \\ y=-x}} \frac{xy}{(x-y)(x^2+y^2)} = 0$$

$$\text{or, } x + y + 3(x + y) \lim_{|x|\to\infty} \frac{-x^2}{2x(2x^2)} + \lim_{|x|\to\infty} \frac{-x^2}{2x(2x^2)} = 0$$

or, $(x + y) + 3(x + y)\cdot 0 + 0 = 0$ or, $x + y = 0$.

Thus, the required asymptotes are $2x - 2y + 3 = 0$ and $x + y = 0$.

Solution (i) Given equation of the curve is $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy - 5y + 6 = 0$.

As the coefficients of x^3 (the highest power term in x) and y^3 (the highest power term in y) are both constant, there does not exist any asymptote parallel to x-axis and y-axis.

To find the oblique asymptotes (of the form $y = mx + c$), we put $x = 1$ and $y = m$ in third and second degree terms of the given equation and obtain

$\phi_3(m) = 1 + 2m - m^2 - 2m^3 = (1 - m^2)(1 + 2m)$ and $\phi_2(m) = 4m^2 + 2m$.

Now $\phi_3(m) = 0 \Rightarrow m = 1, -1, -\dfrac{1}{2}$ (the slopes of the required asymptotes)

and $\phi'_3(m) = 2 - 2m - 6m^2$.

For $m = 1$, $\phi'_3(1) = 2 - 2 - 6 = -6$ and $c = -\dfrac{\phi_2(1)}{\phi'_3(1)} = -\dfrac{6}{-6} = 1$,

For $m = -1$, $\phi'_3(-1) = -2$ and $c = -\dfrac{\phi_2(-1)}{\phi'_3(-1)} = -\dfrac{2}{-2} = 1$ and

For $m = -\frac{1}{2}$, $\phi'_3\left(-\dfrac{1}{2}\right) = \dfrac{3}{2}$ and $c = -\dfrac{\phi_2\left(-\frac{1}{2}\right)}{\phi'_3\left(-\frac{1}{2}\right)} = -\dfrac{0}{\frac{3}{2}} = 0$.

Hence the equations of the required asymptotes are

$y = x + 1$, $y = -x + 1$, $y = -\dfrac{x}{2}$, *i.e.*,

$\mathbf{x + 2y = 0, x + y - 1 = 0}$ **and** $\mathbf{x - y + 1 = 0}$.

(ii) Given equation of the curve is $3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0$.

Clearly, there are no asymptotes parallel to axis...

Thus, the equations of the required asymptotes are $x + 3 = 0, x + 2y - 6 = 0$ and $x - 2y = 0$.

(iv) Given equation of the curve is

$(x + y)^3(x - y)^2 - 2(x + y)^2(x - y)^2 - 2(x^2 + y^2)(x + y) + 2(x - y)^2 + 4(x - y) = 0.$

This is a fifth degree polynomial equation. So it has at most five asymptotes. The asymptotes parallel to $x + y = 0$ are given by

$$(x+y)^3 - 2(x+y)^2 \lim_{\substack{|x|\to\infty \\ \frac{y}{x}=-1}} \frac{(x-y)^2}{(x-y)^2} - 2(x+y) \lim_{\substack{|x|\to\infty \\ \frac{y}{x}=-1}} \frac{x^2+y^2}{(x-y)^2} + \lim_{\substack{|x|\to\infty \\ \frac{y}{x}=-1}} \frac{2(x-y)^2+4(x-y)}{(x-y)^2} = 0$$

$$\text{or, } (x+y)^3 - 2(x+y)^2 - 2(x+y) \lim_{\substack{|x|\to\infty \\ \frac{y}{x}=-1}} \frac{1+\left(\frac{y}{x}\right)^2}{\left(1-\frac{y}{x}\right)^2} + \lim_{\substack{|x|\to\infty \\ \frac{y}{x}=-1}} \frac{2\left(1-\frac{y}{x}\right)^2+4\left(\frac{1}{x}-\frac{1}{x}\cdot\frac{y}{x}\right)}{\left(1-\frac{y}{x}\right)^2} = 0$$

$$\text{or, } (x+y)^3 - 2(x+y)^2 - 2(x+y) \times \frac{2}{4} + \frac{2\times4+4\times0}{4} = 0$$

or, $(x + y)^3 - 2(x + y)^2 - (x + y) + 2 = 0$

or, $a^3 - 2a^2 - a + 2 = 0 \qquad (a = x + y)$

or, $(a - 1)(a + 1)(a - 2) = 0$

or, $(x + y - 1)(x + y + 1)(x + y - 2) = 0$

Thus the asymptotes parallel to $x + y = 0$ are given by $x + y \pm 1 = 0, x + y - 2 = 0$.

Now asymptotes parallel to $x - y = 0$ are given by

$$(x-y)^2 - 2(x-y)^2 \lim_{\substack{|x|\to\infty \\ \frac{y}{x}=1}} \frac{1}{x+y} - 2 \lim_{\substack{|x|\to\infty \\ \frac{y}{x}=1}} \frac{x^2+y^2}{(x+y)^2} + \left[2(x-y)^2+4(x-y)\right] \times \lim_{\substack{|x|\to\infty \\ \frac{y}{x}=1}} \frac{1}{(x+y)^3} = 0$$

$$\text{or, } (x-y)^2 - 0 - 2 \times \frac{(1+1^2)}{(1+1)^2} + 0 = 0$$

or, $(x - y)^2 - 1 = 0$ or, $x - y = \pm 1$

Therefore, all the required asymptotes are given by

$x + y \pm 1 = 0, x + y - 2 = 0, x - y = \pm 1$.

12. Find the asymptotes of the following